Homework 1

(Due on Wednesday before 09:50. Please submit your completed assignment to a TA by email or in-person.)

AI policy. The use of AI tools to complete the assignment is not permitted.

Please complete the questions below using the concepts and techniques introduced in the lectures.

1. *(3 points)* Prove the following theorems of probability from the basic axioms presented in the lecture. For the sake of simplicity, assume that all variables are binary variables.

   - Prove the **law of negation**: for all A,
   $$P(\neg A) = 1 - P(A).$$
   - Prove the **law of total probability**: for all A and B,
   $$P(A) = P(A \mid B)P(B) + P(A \mid \neg B)P(\neg B).$$
   - Prove the **law of disjunction**: for all A and B,
   $$P(A \vee B) = P(A) + P(B) - P(A \wedge B).$$

   [*Hint*: it may be useful to use the fact that '$A \vee B$' $= (A \wedge \neg B) \vee (B \wedge \neg A) \vee (A \wedge B)$, and then to also show that $P(A \wedge \neg B) = P(A) - P(A \wedge B)$.]

2. *(2 points)* Suppose an SCM has exogenous variables A, B, U_C, U_{ABC} with exogenous probabilities $P(A)$, $P(B)$, $P(U_C)$, $P(U_{ABC})$, and endogenous variable C with structural equation:
$$C := (A \wedge B \wedge U_{ABC}) \vee U_C$$
   Write the Causal Bayes Net that describes the system, assuming that U_{ABC} and U_C are not observable (so the only variables in the model are A, B, and C — you will still need to use $P(U_C)$ and $P(U_{ABC})$ to fill the conditional probability table).

3. *(1 point)* Suppose we have the following causal model:
$$C := f_1(A, B)$$
$$D := f_2(C)$$
$$E := f_3(D, A)$$
   where f_n is some function (it doesn't matter which). Write the factorization of the joint distribution $P(A, B, C, D, E)$ defined by the causal model.

4. *(1 point)* Create a causal model where $P(A \mid \mathrm{Do}(B)) \neq P(A \mid B)$, but where B is not a descendant of A. (B is a descendant of A if there is a series of $\rightarrow$ in the DAG of the causal model that lead from A to B).

5. *(2 points)* In lecture we saw the **law of marginalization**, which says that:

$$P(A) = P(A, B) + P(A, \neg B).$$

Prove that the law of marginalization can be extended to conditional probabilities such that:

$$P(A \mid C) = P(A, B \mid C) + P(A, \neg B \mid C).$$

[*Hint*: use the fact that $A \wedge C = (A \wedge C \wedge B) \vee (A \wedge C \wedge \neg B)$.]

6. *(1 point)* Suppose you have a causal model with DAG $C \rightarrow A \leftarrow B$. In lecture we saw that

$$P(A \mid C) = P(A \mid B, C)P(B) + P(A \mid \neg B, C)P(\neg B).$$

Prove this fact. [*Hint*: use the result derived in the previous exercise, and the factorization induced by the causal model.]